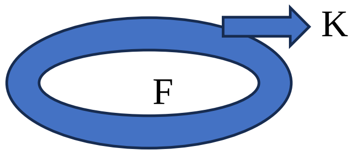
FIELDS

SEC 5.1 EXTENSION FIELDS

DEFINITION:

Let F be a field ; a field K is said to be an ***extension*** of F if K contained F(K $\supset$ F).



NOTE:- (1). Kis an extension field of F if Fis a subfield of K.

(2).If K is an extension field of F ,it can be treated as vector space over F and dimension of V.Sp. taken as degree of the field.

Definition: The ***degree of K*** over F is the dimension of K as a vector space over F, denoted by [K:F].

Definition: If K is a finite dimension vector space over F, then K is a **finite extension** of F.

THEOREM 5.1.1:-

If L is a f.extn. of K and if K is a f.extn. of F , then L is a f.extn. of F .

Moreover , [L:F] = [L:K] [K:F]

PROOF:-

Suppose [L:K] = m and [K:F] =n. ---------------(1)

Let $v_1, v_2, \dots, v_m$ be a basis of L over K and

let $w_1, w_2, \dots, w_n$ be a basis of K over F. -------------(2)

Consider any **element t ϵ L**, then t can be written as linear combination of basis vectors $v_1, v_2, \dots, v_m$.

i.e., t = $k_1 v_1 + \cdots + k_m v_m$, where $k_i \in K$ -----------(3)

Since K is a vector space over F and k_i's ϵ K ,each k_i can be written as l.c. of $w_1, w_2, \dots, w_n$.

i.e., $k_1 = \alpha_{11} w_1 + \cdots + \alpha_{1n} w_n$.

$k_2 = \alpha_{21} w_1 + \cdots + \alpha_{2n} w_n$.

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. $k_m = \alpha_{m1} w_1 + \cdots + \alpha_{mn} w_n$.,where $\alpha_{ij} \in F$ (4)

Substituting these values of k_i 's in (3) ,

t = $(\alpha_{11}w_1 + \cdots + \alpha_{1n}w_n)\, v_1$ + …+ $(\alpha_{m1}w_1 + \cdots + \alpha_{mn}w_n)\, v_m$.

i.e.,t $= \alpha_{11}v_1w_1 + \cdots + \alpha_{1n}v_1w_n + \cdots + \alpha_{m1}v_mw_1 + \cdots + \alpha_{mn}v_mw_n$.

Thus t is a l.c of elements v_iw_j over F.So **v_iw_j Span L over F.----(A)**

Now we can prove v_iw_j are linearly independent over F.

Suppose $f_{11}v_1w_1 + \cdots + f_{1n}v_1w_n + f_{21}v_2w_1 + \cdots + f_{2n}v_2w_n$

+…+$f_{m1}v_mw_1 + \cdots + f_{mn}v_mw_n$= 0--------(5)

Claim: Each f_{ij} = 0.,so that the mn elements v_iw_j are linearly independent.

For, regrouping the terms in (5),$k_1v_1 + \cdots + k_mv_m = 0$, .------(6)

Where $k_i = f_{i1}w_1 + \cdots + f_{in}w_n$, i =1,…,m

⟹ Each k_i =0.(since$v_1, v_2, \ldots, v_m$ be a basis of L over K)

⟹ $k_i = f_{i1}w_1 + \cdots + f_{in}w_n = 0$, for i =1,…,m .

=>**each f_{ij} = 0**(since $w_1, w_2, \ldots, w_n$ be a basis of K over F).

So v_iw_j are linearly independent over F.---(B)

Thus v_iw_j (mn) elements form a basis of L over F.(From (A)& (B))

Thus **[L:F] =mn. Hence L is a f.extn. of F.//**

Moreover[L:F] = mn

i.e.,[L:F] = [L:K][K:F](by eqn(1)//

Hence the theorem.

COR :1

If L is a f.extn of F and if K is a subfield of L that contains F,

then [K:F] | [L:F]

PROOF: Suppose L,K,F are fields s.t L⊃K⊃F and [L:F] is finite.

Any element in L is linearly independent over K,are all linearly independent over F.Thus [L:F] is finite =>[L:K] is finite.

Since K is a subspace of L,[K:F] is also finite.

By thm/. [L:F] = mn, where [K:F] =n. Thus n|nm. i.e., [K:F] | [L: F].

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Note: If [L:F] is a prime number, then there can be no fields properly between L and F.

DEFINITION: An element ***a* ∈ *K*** is said to be ***algebraic*** over F if there exist elements $\alpha_0 , \alpha_1 , \dots , \alpha_n$ in F , not all 0, such that $\alpha_0 a^n + \alpha_1 a^{n-1} + \cdots + \alpha_n = 0$.

Note: 1.Consider the ring of Polynomials in x over F denoted by F[x].If q(x) ϵ F[x] and q(x) = $\beta_0 x^n + \beta_1 x^{n-1} + \cdots + \beta_n$,then any element b ϵ K ,q(b) can be written as

q(b) = $\beta_0 b^n + \beta_1 b^{n-1} + \cdots + \beta_n$ ϵ K.

i.e., q(b) is equal to q(x) at x= b.

2. In these terms ***a* ∈ *K*** is said to be ***algebraic*** over F if there is a nonzero polynomial p(x) ϵ F[x],so that a satisfies p(x).

i.e., p(a) = 0.

Definition: Adjoining an element *a* ∈ *K to F.*

Let K be an extension of F and let ***a* ∈ *K*.**Consider the collection $\mathscr{M}$ of all subfield of K which contain both F and a. Since K itself is a subfield of K, an element of $\mathscr{M}$ and so is nonempty. The intersection of all these subfields of K which are members of $\mathscr{M}$ is a subfield of K. That subfield is denoted by F(a).

It has the following properties:-

(i). It contains both F and a

(ii).F(a) is in $\mathscr{M}$

Thus F(a) is the smallest subfield of K containing both F and a. So F(a) called as subfield obtained by **adjoining a to F.**

THEOREM 5.1.2 : An element *a* ∈ *K is algebraic over F if and only if F(a) is a finite extension of F.*

PROOF: Sufficient part:

Suppose that ***F(a) is a f.extn. of F*** and that [F(a):F] = m.------(1)

Consider the elements $1, a, a^2, \dots, a^m$; they are all in F(a)and are m+1 in number, greater than the degree ,they are linearly dependent over F.

Therefore ,there are elements $\alpha_0 , \alpha_1 , \dots , \alpha_n$ in F ,not all 0,such that $\alpha_0 1 + \alpha_1 a + \cdots + \alpha_m a^m = 0$. ----------(2)

Hence ***a is algebraic over F*** and satisfies the nonzero polynomial

p(x)=$\alpha_0 + \alpha_1 x + \cdots + \alpha_m x^m$ in F[x] .

Thus second part proved.//

Conversely, Suppose ***a ∈ K is algebraic over F*** .By the definition ,there exist a nonzero polynomial p(x) in F[x] such that p(a)=0.Suppose that deg p(x) is the smallest one.

Claim: p(x) is irreducible over F

For, suppose that p(x) = f(x).g(x),where f(x),g(x) in F[x].--------(1)

If p(a) = 0 $\Rightarrow$ f(a)g(a) = 0

$\Rightarrow$ f(a) = 0 or g(a) = 0

Also deg f(x) ≥ deg p(x) (or)

deg g(x) ≥ deg p(x) ----------------(2)(∵deg p(x) is smallest)

But from eqn/. (1),deg p(x) = deg { f(x).g(x)}

= deg f(x) + deg g(x)------(3)

From (2) &(3) we conclude either deg f(x) = 0 (or) deg g(x) = 0

$\Rightarrow$ Either f(x) or g(x) is a constant polynomial.

∴p(x) is irreducible over F.

Consider the mapping Ψ: F[x] ⟶ F(a) defined as Ψ(h(x)) = h(a) is a ring homomorphism..

Let V denote the kernel of Ψ.By defn/. V = {h(x) ϵ F(x)| h(a) = 0}.

Also p(x) ϵ V (since p(a) = 0). Since p(x) is of smallest degree ,h(x) in V is a multiple of p(x).

By division algorithm we can find q(x) , r(x) ∈F[x], so that h(x) = q(x).p(x) +r(x), where either r(x) = 0 or deg r(x) ≤ deg p(x).

Claim : r(x) = 0.

For ,otherwise r(x) = h(x) – q(x).p(x).

r(a) = h(a) -q(a).p(a) = 0 {∵ h(a) = 0,p(a) = 0

$\Rightarrow$ r(x) ϵ V and deg r(x) ≤ deg p(x) is a contradiction.

Thus r(x) = 0.

Therefore h(x) is a multiple of p(x).

Since F[x] is an Euclidean ring and V consists of all element which are multiple of p(x) ,V is an ideal of F[x],it is also a maximal ideal.(∵ p(x) is irreducible over F).

By general homomorphism on rings ,$\frac{F[x]}{V}$ is isomorphic to the image of F[x] under Ψ.

The image of F[x] under Ψ is a subfield of F(a) which contains both F and a.

But by the defn/. F(a) is the smallest subfield of K that contains both F and a.

∴ The image of F[x] under Ψ is all of F(a).

Hence $\frac{F[x]}{V}$ **is isomorphic to F(a) itself**. Then dim F(a) is same as dim $\frac{F[x]}{V}$.

Since V is generated by p(x) , dim $\frac{F[x]}{V}$ is same as deg p(x),say n.

∴ [F(a]:F] = deg p(x) = n.

i.e., ***F(a) is a finite extension of F.//.-------THM/. Proved.***

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DEFINITION: Algebraic of degree n.

The element ***a ∈ K*** is said to be ***algebraic of degree n*** over F if it satisfies a nonzero polynomial over F of degree n but no nonzero polynomial of lower degree.

THEOREM:5.1.3

If ***a ∈ K*** is said to be ***algebraic of degree n*** over F, then [F(a):F] = n.

Proof: Converse part of the theorem 5.1.2.

THEOREM :5.1.4

If a, b ϵ K are algebraic over F, then a ± b, ab and a / b (if b ≠ 0)are all algebraic over F. In other words, the elements in K which are algebraic over F, form a subfield of K.

PROOF:

Suppose that a is algebraic of degree m over F and b is algebraic of degree n over F, then by theorem [F(a):F] = m & [F(b):F] = n -------------(1)

Now ,let T = F(a) the subfield of K is of degree m over F.-------(2)

Now b is algebraic of degree n over F.⟹that it is algebraic of degree atmost n over over T which contains F.Thus the subfield W = T(b) of K is of degree atmost n over T. i.e., [F(a):F] =m &[T(b):T] ≤ n

Now [W:F] = [W:T] [T:F]

≤ nm.

$Thus\ W\ is\ a\ finite\ extension\ of\ F.$

Since a , b ϵ W, a ± b, ab and a / b ϵ W. Then by thm/. a ± b, ab and a / b are all algebraic over F.

Hence the thm//..

Note: Since [W:F] ≤ nm, every element in W satisfies a polynomial of degree at most mn.

Cor: If a, b ϵ K are algebraic over F of degree m & n respectively, then a $\pm$ b, ab and a / b are all algebraic over F of degree at most mn.

Notations:

In the thm/. ,named two extensions F(a) as T and T(b) as W. Thus W= (F(a))(b)

= F(a,b)

Similarly (F(b))(a) = F(b,a). Also F(a,b) =F(b,a) for all a,b ϵ K.

Continuing this pattern, we can define F(a_1,a_2,…,a_n) for all elements a_1,a_2,…,a_n ϵ K are algebraic over F.

DEFINITION:-

The extension K of F is called ***algebraic extension*** of F if every element in K is algebraic over F.

THEOREM:5.1.5

If Lis an algebraic extension of K and if K is an algebraic extension of F then L is an algebraic extension of F .

Proof: Consider an arbitrary element u in L.

Since ***L is an algebraic extension of K*** ,there exist a nonzero polynomial of the form

$x^n + \sigma_1 x^{n-1} + \cdots + \sigma_n x^0$,where $\sigma_1, \ldots \sigma_n$ are in K.

Since ***K is algebraic extension of F,*** each σ_i is algebraic over F, We have finite extensions F(σ_1) ,...,F($\sigma_1, \ldots \sigma_n$) by thm/(5.1.3).

Let M= F($\sigma_1, \ldots \sigma_n$) is a finite extension of F, by defn/. σ_i ϵ M.

Since u satisfies the polynomial $x^n + \sigma_1 x^{n-1} + \cdots + \sigma_n$ whose coefficients are in M ,u is algebraic over M.

Then M(u) is a finite extension of M. (by thm./.5.1.2)

Therefore by Thm/(5.1.1). [M(u):F] = [M(u):M][M:F],

⟹ M(u) is the finite extension of F.

⟹ u is algebraic over F. Thus ***L is algebraic extn/. of F.***

Hence the THM//.

Definition: A complex number is said to be an ***algebraic number*** if it is algebraic over the field of rational numbers. Otherwise called ***transcendental.***

Problem:1. What is the degree of $\sqrt{2}$ + $\sqrt{3}$ over Q ?

Solution: Let $x = \sqrt{2}$ +$\sqrt{3}$ ------------(1)

Squaring on both sides,

x^2 =($\sqrt{2}$ + $\sqrt{3}$)^2 ,

i.e., x^2 = 5+ 2 $\sqrt{2}$ $\sqrt{3}$

i.e., x^2 - 5 = 2 $\sqrt{6}$

Squaring on both sides,

$(x^2 - 5)^2$ = 24

i.e., $x^4 - 10x^2 + 1 = 0.$

We have a polynomial of degree 4.$\Longrightarrow$ **The degree of $\sqrt{2}$ + $\sqrt{3}$ over Q is 4.//**

2. What is the degree of $\sqrt{2}$ $\sqrt{3}$ over Q ? Home Work.